

Induced differential forms on manifolds of functions

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Abstract

Differential forms on the Fréchet manifold $\mathcal{F}(S, M)$ of smooth functions on a compact k -dimensional manifold S can be obtained in a natural way from pairs of differential forms on M and S by the hat pairing. Special cases are the transgression map $\Omega^p(M) \rightarrow \Omega^{p-k}(\mathcal{F}(S, M))$ (hat pairing with a constant function) and the bar map $\Omega^p(M) \rightarrow \Omega^p(\mathcal{F}(S, M))$ (hat pairing with a volume form). We develop a hat calculus similar to the tilda calculus for non-linear Grassmannians [HV04].

1 Introduction

Pairs of differential forms on the finite dimensional manifolds M and S induce differential forms on the Fréchet manifold $\mathcal{F}(S, M)$ of smooth functions. More precisely, if S is a compact oriented k -dimensional manifold, the hat pairing is:

$$\Omega^p(M) \times \Omega^q(S) \rightarrow \Omega^{p+q-k}(\mathcal{F}(S, M))$$

$$\widehat{\omega \cdot \alpha} = \int_S \text{ev}^* \omega \wedge \text{pr}^* \alpha,$$

where $\text{ev} : S \times \mathcal{F}(S, M) \rightarrow M$ denotes the evaluation map, $\text{pr} : S \times \mathcal{F}(S, M) \rightarrow S$ the projection and \int_S fiber integration. We show that the hat pairing is compatible with the canonical $\text{Diff}(M)$ and $\text{Diff}(S)$ actions on $\mathcal{F}(S, M)$, and with the exterior derivative. As a consequence we obtain a hat pairing in cohomology.

The hat (transgression) map is the hat pairing with the constant function 1, so it associates to any form $\omega \in \Omega^p(M)$ the form $\widehat{\omega \cdot 1} = \widehat{\omega} = \int_S \text{ev}^* \omega \in \Omega^{p-k}(\mathcal{F}(S, M))$. Since $\mathfrak{X}(M)$ acts infinitesimally transitive on the open subset $\text{Emb}(S, M) \subset \mathcal{F}(S, M)$ of embeddings of the k -dimensional oriented manifold S into M [H76], the expression of $\widehat{\omega}$ at $f \in \text{Emb}(S, M)$ is

$$\widehat{\omega}(X_1 \circ f, \dots, X_{p-k} \circ f) = \int_S f^*(i_{X_{p-k}} \dots i_{X_1} \omega), \quad X_1, \dots, X_{p-k} \in \mathfrak{X}(M).$$

When S is the circle, then one obtains the usual transgression map with values in the space of $(p-1)$ -forms on the free loop space of M .

Let $\text{Gr}_k(M)$ be the non-linear Grassmannian of k -dimensional oriented submanifolds of M . The tilda map associates to every $\omega \in \Omega^p(M)$ a differential $(p-k)$ -form on $\text{Gr}_k(M)$ given by [HV04]

$$\tilde{\omega}(\tilde{Y}_N^1, \dots, \tilde{Y}_N^{p-k}) = \int_N i_{Y_N^{p-k}} \cdots i_{Y_N^1} \omega, \quad \forall \tilde{Y}_N^1, \dots, \tilde{Y}_N^{p-k} \in \Gamma(TN^\perp) = T_N \text{Gr}_k(M),$$

for \tilde{Y}_N section of the orthogonal bundle TN^\perp represented by the section Y_N of $TM|_N$. The natural map

$$\pi : \text{Emb}(S, M) \rightarrow \text{Gr}_k(M), \quad \pi(f) = f(S)$$

provides a principal bundle with the group $\text{Diff}_+(S)$ of orientation preserving diffeomorphisms of S as structure group.

The hat map on $\text{Emb}(S, M)$ and the tilda map on $\text{Gr}_k(M)$ are related by $\hat{\omega} = \pi^* \tilde{\omega}$. This is the reason why for the hat calculus one has similar properties to those for the tilda calculus. The tilda calculus was used to study the non-linear Grassmannian of co-dimension two submanifolds as symplectic manifold [HV04]. We apply the hat calculus to the hamiltonian formalism for p -branes and open p -branes [AS05] [BZ05].

The bar map $\bar{\omega} = \widehat{\omega \cdot \mu}$ is the hat pairing with a fixed volume form μ on S , so

$$\bar{\omega}(Y_f^1, \dots, Y_f^p) = \int_S \omega(Y_f^1, \dots, Y_f^p) \mu, \quad \forall Y_f^1, \dots, Y_f^p \in \Gamma(f^*TM) = T_f \mathcal{F}(S, M).$$

We use the bar calculus to study $\mathcal{F}(S, M)$ with symplectic form $\bar{\omega}$ induced by a symplectic form ω on M . The natural actions of $\text{Diff}_{ham}(M, \omega)$ and $\text{Diff}_{ex}(S, \mu)$, the group of hamiltonian diffeomorphisms of M and the group of exact volume preserving diffeomorphisms of S , are two commuting hamiltonian actions on $\mathcal{F}(S, M)$. Their momentum maps form the dual pair for ideal incompressible fluid flow [MW83] [GBV09].

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2 Hat pairing

We denote by $\mathcal{F}(S, M)$ the set of smooth functions from a compact oriented k -dimensional manifold S to a manifold M . It is a Fréchet manifold in a natural way [KM97]. Tangent vectors at $f \in \mathcal{F}(S, M)$ are identified with vector fields on M along f , i.e. sections of the pull-back vector bundle f^*TM .

Let $\text{ev} : S \times \mathcal{F}(S, M) \rightarrow M$ be the evaluation map $\text{ev}(x, f) = f(x)$ and $\text{pr} : S \times \mathcal{F}(S, M) \rightarrow S$ the projection $\text{pr}(x, f) = x$. A pair of differential forms $\omega \in \Omega^p(M)$ and $\alpha \in \Omega^q(S)$ determines a differential form $\widehat{\omega \cdot \alpha}$ on $\mathcal{F}(S, M)$ by the

fiber integral over S (whose definition and properties are listed in the appendix) of the $(p+q)$ -form $\text{ev}^* \omega \wedge \text{pr}^* \alpha$ on $S \times \mathcal{F}(S, M)$:

$$\widehat{\omega \cdot \alpha} = \int_S \text{ev}^* \omega \wedge \text{pr}^* \alpha \quad (1)$$

In this way we obtain a bilinear map called the *hat pairing*:

$$\Omega^p(M) \times \Omega^q(S) \rightarrow \Omega^{p+q-k}(\mathcal{F}(S, M)).$$

An explicit expression of the hat pairing avoiding fiber integration is:

$$(\widehat{\omega \cdot \alpha})_f(Y_f^1, \dots, Y_f^{p+q-k}) = \int_S f^*(i_{Y_f^{p+q-k}} \dots i_{Y_f^1}(\omega \circ f)) \wedge \alpha, \quad (2)$$

for $Y_f^1, \dots, Y_f^{p+q-k}$ vector fields on M along $f \in \mathcal{F}(S, M)$. Here we denote by $f^* \beta_f$ the "restricted pull-back" by f of a section β_f of $f^*(\Lambda^m T^* M)$, which is a differential m -form on S given by $f^* \beta_f : x \in S \mapsto (\Lambda^m T_x^* f)(\beta_f(x)) \in \Lambda^m T_x^* S$, where $T_x^* f : T_x^* M \rightarrow T_x^* S$ denotes the dual of $T_x f$.

The fact that (1) and (2) provide the same differential form on $\mathcal{F}(S, M)$ can be deduced from the identity

$$(\text{ev}^* \omega)_{(x,f)}(Y_f^1, \dots, Y_f^{p-k}, X_x^1, \dots, X_x^k) = f^*(i_{Y_f^{p-k}} \dots i_{Y_f^1}(\omega \circ f))(X_x^1, \dots, X_x^k)$$

for $Y_f^1, \dots, Y_f^{p-k} \in T_f \mathcal{F}(S, M)$ and $X_x^1, \dots, X_x^k \in T_x S$.

Since $\mathfrak{X}(M)$ acts infinitesimally transitive on the open subset $\text{Emb}(S, M) \subset \mathcal{F}(S, M)$ of embeddings of the k -dimensional oriented manifold S into M , we express $\widehat{\omega}$ at $f \in \text{Emb}(S, M)$ as:

$$(\widehat{\omega \cdot \alpha})_f(X_1 \circ f, \dots, X_{p+q-k} \circ f) = \int_S f^*(i_{X_{p+q-k}} \dots i_{X_1} \omega) \wedge \alpha. \quad (3)$$

One uses the fact that the "restricted pull-back" by f of $i_{X_{p+q-k} \circ f} \dots i_{X_1 \circ f}(\omega \circ f)$ is $f^*(i_{X_{p+q-k}} \dots i_{X_1} \omega)$.

Next we show that the hat pairing is compatible with the exterior derivative of differential forms.

Theorem 1. *The exterior derivative \mathbf{d} is a derivation for the hat pairing, i.e.*

$$\mathbf{d}(\widehat{\omega \cdot \alpha}) = (\widehat{\mathbf{d}\omega}) \cdot \alpha + (-1)^p \widehat{\omega \cdot \mathbf{d}\alpha}, \quad (4)$$

where $\omega \in \Omega^p(M)$ and $\alpha \in \Omega^q(S)$.

Proof. Differentiation and fiber integration along the boundary free manifold S commute, so

$$\begin{aligned} \mathbf{d}(\widehat{\omega \cdot \alpha}) &= \mathbf{d} \int_S \text{ev}^* \omega \wedge \text{pr}^* \alpha = \int_S \mathbf{d}(\text{ev}^* \omega \wedge \text{pr}^* \alpha) \\ &= \int_S \text{ev}^* \mathbf{d}\omega \wedge \text{pr}^* \alpha + (-1)^p \int_S \text{ev}^* \omega \wedge \text{pr}^* \mathbf{d}\alpha = (\widehat{\mathbf{d}\omega}) \cdot \alpha + (-1)^p \widehat{\omega \cdot \mathbf{d}\alpha} \end{aligned}$$

for all $\omega \in \Omega^p(M)$ and $\alpha \in \Omega^q(S)$. \square

The differential form $\widehat{\omega \cdot \alpha}$ is exact if ω is closed and α exact (or if α is closed and ω exact). In the special case $p+q = k$ these conditions imply that the function $\widehat{\omega \cdot \alpha}$ on $\mathcal{F}(S, M)$ vanishes.

Corollary 2. *The hat pairing induces a bilinear map on de Rham cohomology spaces*

$$H^p(M) \times H^q(S) \rightarrow H^{p+q-k}(\mathcal{F}(S, M)). \quad (5)$$

In particular there is a bilinear map

$$H^p(M) \times H^q(M) \rightarrow H^{p+q-k}(\text{Diff}(M)).$$

Remark 3. The cohomology group $H^q(S)$ is isomorphic to the homology group $H_{k-q}(S)$ by Poincaré duality. With the notation $n = k - q$, the hat pairing (5) becomes

$$H^p(M) \times H_n(S) \rightarrow H^{p-n}(\mathcal{F}(S, M)),$$

and it is induced by the map $(\omega, \sigma) \mapsto \int_{\sigma} \text{ev}^* \omega$, for differential p -forms ω on M and n -chains σ on S .

If S is a manifold with boundary, then formula (4) receives an extra term coming from integration over the boundary. Let $i_{\partial} : \partial S \rightarrow S$ be the inclusion and $r_{\partial} : \mathcal{F}(S, M) \rightarrow \mathcal{F}(\partial S, M)$ the restriction map.

Proposition 4. *The identity*

$$\mathbf{d}(\widehat{\omega \cdot \alpha}) = (\widehat{\mathbf{d}\omega}) \cdot \alpha + (-1)^p \widehat{\omega \cdot \mathbf{d}\alpha} + (-1)^{p+q-k} r_{\partial}^*(\widehat{\omega \cdot i_{\partial}^* \alpha}^{\partial}) \quad (6)$$

holds for $\omega \in \Omega^p(M)$ and $\alpha \in \Omega^q(S)$, where the upper index ∂ assigned to the hat means the pairing

$$\Omega^p(M) \times \Omega^q(\partial S) \rightarrow \Omega^{p+q-k+1}(\mathcal{F}(\partial S, M)).$$

Proof. For any differential n -form β on $S \times \mathcal{F}(S, M)$, the identity

$$\mathbf{d} \int_S \beta - \int_S \mathbf{d}\beta = (-1)^{n-k} \int_{\partial S} (i_{\partial} \times 1_{\mathcal{F}(S, M)})^* \beta$$

holds because of the identity (19) from the appendix. The obvious formulas

$$\text{pr} \circ (i_{\partial} \times 1_{\mathcal{F}(S, M)}) = i_{\partial} \circ \text{pr}_{\partial}, \quad \text{ev} \circ (i_{\partial} \times 1_{\mathcal{F}(S, M)}) = \text{ev}_{\partial},$$

for $\text{ev}_{\partial} : \partial S \times \mathcal{F}(S, M) \rightarrow M$ and $\text{pr}_{\partial} : \partial S \times \mathcal{F}(S, M) \rightarrow \partial S$, are used to compute

$$\begin{aligned} \mathbf{d}(\widehat{\omega \cdot \alpha}) &= \mathbf{d} \int_S \text{ev}^* \omega \wedge \text{pr}^* \alpha \\ &= \int_S \mathbf{d}(\text{ev}^* \omega \wedge \text{pr}^* \alpha) + (-1)^{p+q-k} \int_{\partial S} (i_{\partial} \times 1_{\mathcal{F}(S, M)})^* (\text{ev}^* \omega \wedge \text{pr}^* \alpha) \\ &= \int_S \text{ev}^* \mathbf{d}\omega \wedge \text{pr}^* \alpha + (-1)^p \int_S \text{ev}^* \omega \wedge \text{pr}^* \mathbf{d}\alpha + (-1)^{p+q-k} \int_{\partial S} \text{ev}_{\partial}^* \omega \wedge \text{pr}_{\partial}^* i_{\partial}^* \alpha \\ &= (\widehat{\mathbf{d}\omega}) \cdot \alpha + (-1)^p \widehat{\omega \cdot \mathbf{d}\alpha} + (-1)^{p+q-k} r_{\partial}^*(\widehat{\omega \cdot i_{\partial}^* \alpha}^{\partial}), \end{aligned}$$

thus obtaining the requested identity. \square

Left $\text{Diff}(M)$ action. The natural left action of the group of diffeomorphisms $\text{Diff}(M)$ on $\mathcal{F}(S, M)$ is $\varphi \cdot f = \varphi \circ f$. The infinitesimal action of $X \in \mathfrak{X}(M)$ is the vector field \bar{X} on $\mathcal{F}(S, M)$:

$$\bar{X}(f) = X \circ f, \quad \forall f \in \mathcal{F}(S, M).$$

We denote by $\bar{\varphi}$ the diffeomorphism of $\mathcal{F}(S, M)$ induced by the action of $\varphi \in \text{Diff}(M)$, so $\bar{\varphi}(f) = \varphi \circ f$ is the push-forward by φ .

Proposition 5. *Given $\omega \in \Omega^p(M)$ and $\alpha \in \Omega^q(S)$, the identity*

$$\bar{\varphi}^* \widehat{\omega \cdot \alpha} = (\widehat{\varphi^* \omega}) \cdot \alpha \quad (7)$$

and its infinitesimal version

$$L_{\bar{X}} \widehat{\omega \cdot \alpha} = (\widehat{L_X \omega}) \cdot \alpha \quad (8)$$

hold for all $\varphi \in \text{Diff}(M)$ and $X \in \mathfrak{X}(M)$.

Proof. Using the expression (1) of the hat pairing and identity (15) from the appendix, we have:

$$\begin{aligned} \bar{\varphi}^* \widehat{\omega \cdot \alpha} &= \bar{\varphi}^* \int_S \text{ev}^* \omega \wedge \text{pr}^* \alpha = \int_S (1_S \times \bar{\varphi})^* (\text{ev}^* \omega \wedge \text{pr}^* \alpha) \\ &= \int_S \text{ev}^* \varphi^* \omega \wedge \text{pr}^* \alpha = (\widehat{\varphi^* \omega}) \cdot \alpha, \end{aligned}$$

since $\text{pr} \circ (1_S \times \bar{\varphi}) = \text{pr}$ and $\text{ev} \circ (1_S \times \bar{\varphi}) = \varphi \circ \text{ev}$. \square

A similar result is obtained for any smooth map $\eta \in \mathcal{F}(M_1, M_2)$ and its push-forward $\bar{\eta} : \mathcal{F}(S, M_1) \rightarrow \mathcal{F}(S, M_2)$, $\bar{\eta}(f) = \eta \circ f$:

$$\bar{\eta}^* \widehat{\omega \cdot \alpha} = \widehat{\eta^* \omega \cdot \alpha},$$

for all $\omega \in \Omega^p(M_2)$ and $\alpha \in \Omega^q(S)$.

Lemma 6. *For all vector fields $X \in \mathfrak{X}(M)$, the identity $i_{\bar{X}} \widehat{\omega \cdot \alpha} = (\widehat{i_X \omega}) \cdot \alpha$ holds.*

Proof. The vector field $0_S \times \bar{X}$ on $S \times \mathcal{F}(S, M)$ is ev-related to the vector field X on M , so

$$\begin{aligned} i_{\bar{X}} \widehat{\omega \cdot \alpha} &= i_{\bar{X}} \int_S \text{ev}^* \omega \wedge \text{pr}^* \alpha = \int_S i_{0_S \times \bar{X}} (\text{ev}^* \omega \wedge \text{pr}^* \alpha) \\ &= \int_S \text{ev}^* (i_X \omega) \wedge \text{pr}^* \alpha = (\widehat{i_X \omega}) \cdot \alpha. \end{aligned}$$

At step two we use formula (18) from the appendix. \square

Right $\text{Diff}(S)$ action. The natural right action of the diffeomorphism group $\text{Diff}(S)$ on $\mathcal{F}(S, M)$ can be transformed into a left action by $\psi \cdot f = f \circ \psi^{-1}$. The infinitesimal action of $Z \in \mathfrak{X}(S)$ is the vector field \widehat{Z} on $\mathcal{F}(S, M)$:

$$\widehat{Z}(f) = -Tf \circ Z, \quad \forall f \in \mathcal{F}(S, M).$$

We denote by $\widehat{\psi}$ the diffeomorphism of $\mathcal{F}(S, M)$ induced by the action of ψ , so $\widehat{\psi}(f) = f \circ \psi^{-1}$ is the pull-back by ψ^{-1} .

Proposition 7. *Given $\omega \in \Omega^p(M)$ and $\alpha \in \Omega^q(S)$, the identity*

$$\widehat{\psi^* \omega \cdot \alpha} = \widehat{\omega \cdot \psi^* \alpha}$$

and its infinitesimal version

$$L_{\widehat{Z}} \widehat{\omega \cdot \alpha} = \widehat{\omega \cdot L_Z \alpha}$$

hold for all orientation preserving $\psi \in \text{Diff}(S)$ and $Z \in \mathfrak{X}(S)$.

Proof. The obvious identities $\text{ev} \circ (1_S \times \widehat{\psi}) = \text{ev} \circ (\psi^{-1} \times 1_{\mathcal{F}})$, $\text{pr} \circ (1_S \times \widehat{\psi}) = \text{pr}$ and $\text{pr} \circ (\psi \times 1_{\mathcal{F}}) = \psi \circ \text{pr}$ are used in the computation

$$\begin{aligned} \widehat{\psi^* \omega \cdot \alpha} &= \widehat{\psi}^* \int_S \text{ev}^* \omega \wedge \text{pr}^* \alpha = \int_S (1_S \times \widehat{\psi})^* (\text{ev}^* \omega \wedge \text{pr}^* \alpha) \\ &= \int_S ((\psi^{-1} \times 1_{\mathcal{F}})^* \text{ev}^* \omega) \wedge \text{pr}^* \alpha = \int_S \text{ev}^* \omega \wedge (\psi \times 1_{\mathcal{F}})^* \text{pr}^* \alpha \\ &= \int_S \text{ev}^* \omega \wedge \text{pr}^* \psi^* \alpha = \widehat{\omega \cdot \psi^* \alpha}, \end{aligned}$$

together with formula (17) from the appendix at step four. \square

Lemma 8. *The identity $i_{\widehat{Z}} \widehat{\omega \cdot \alpha} = (-1)^p \widehat{\omega \cdot i_Z \alpha}$ holds for all vector fields $Z \in \mathfrak{X}(S)$, if $\omega \in \Omega^p(M)$.*

Proof. The infinitesimal version of the first identity in the proof of proposition 7 is $T \text{ev} \cdot (0_S \times \widehat{Z}) = T \text{ev} \cdot (-Z \times 0_{\mathcal{F}(S, M)})$, so we compute:

$$\begin{aligned} i_{\widehat{Z}} \widehat{\omega \cdot \alpha} &= i_{\widehat{Z}} \int_S \text{ev}^* \omega \wedge \text{pr}^* \alpha = \int_S i_{0_S \times \widehat{Z}} (\text{ev}^* \omega \wedge \text{pr}^* \alpha) \\ &= \int_S (i_{0_S \times \widehat{Z}} \text{ev}^* \omega) \wedge \text{pr}^* \alpha = \int_S (i_{-Z \times 0_{\mathcal{F}(S, M)}} \text{ev}^* \omega) \wedge \text{pr}^* \alpha \\ &= \int_S i_{-Z \times 0_{\mathcal{F}(S, M)}} (\text{ev}^* \omega \wedge \text{pr}^* \alpha) - \int_S (-1)^p \text{ev}^* \omega \wedge i_{-Z \times 0_{\mathcal{F}(S, M)}} \text{pr}^* \alpha \\ &= (-1)^p \int_S \text{ev}^* \omega \wedge \text{pr}^* (i_Z \alpha) = (-1)^p \widehat{\omega \cdot i_Z \alpha}. \end{aligned}$$

At step two we use formula (18) from the appendix. \square

3 Tilda map and hat map

Let $\text{Gr}_k(M)$ be the non-linear Grassmannian (or differentiable Chow variety) of compact oriented k -dimensional submanifolds of M . It is a Fréchet manifold [KM97] and the tangent space at $N \in \text{Gr}_k(M)$ can be identified with the space of smooth sections of the normal bundle $TN^\perp = (TM|_N)/TN$. The tangent vector at N determined by the section $Y_N \in \Gamma(TM|_N)$ is denoted by $\tilde{Y}_N \in T_N \text{Gr}_k(M)$.

The *tilda map* [HV04] associates to any p -form ω on M a $(p-k)$ -form $\tilde{\omega}$ on $\text{Gr}_k(M)$ by:

$$\tilde{\omega}_N(\tilde{Y}_N^1, \dots, \tilde{Y}_N^{p-k}) = \int_N i_{Y_N^{p-k}} \cdots i_{Y_N^1} \omega. \quad (9)$$

Here all \tilde{Y}_N^j are tangent vectors at $N \in \text{Gr}_k(M)$, i.e. sections of TN^\perp represented by sections Y_N^j of $TM|_N$. Then $i_{Y_N^{p-k}} \cdots i_{Y_N^1} \omega \in \Omega^k(N)$ does not depend on representatives Y_N^j of \tilde{Y}_N^j , and integration is well defined since $N \in \text{Gr}_k(M)$ comes with an orientation.

Let S be a compact oriented k -dimensional manifold. The *hat map* is the hat pairing with the constant function $1 \in \Omega^0(S)$. It associates to any form $\omega \in \Omega^p(M)$ the form $\hat{\omega} \in \Omega^{p-k}(\mathcal{F}(S, M))$:

$$\hat{\omega} = \widehat{\omega \cdot 1} = \int_S \text{ev}^* \omega. \quad (10)$$

On the open subset $\text{Emb}(S, M) \subset \mathcal{F}(S, M)$ of embeddings, formula (2) gives

$$\hat{\omega}(X_1 \circ f, \dots, X_{p-k} \circ f) = \int_S f^*(i_{X_{p-k}} \cdots i_{X_1} \omega). \quad (11)$$

Remark 9. The hat map induces a transgression on cohomology spaces

$$H^p(M) \rightarrow H^{p-k}(\mathcal{F}(S, M)).$$

When S is the circle, then one obtains the usual transgression map with values in the $(p-1)$ -th cohomology space of the free loop space of M .

Let π denote the natural map

$$\pi : \text{Emb}(S, M) \rightarrow \text{Gr}_k(M), \quad \pi(f) = f(S).$$

where the orientation on $f(S)$ is chosen such that the diffeomorphism $f : S \rightarrow f(S)$ is orientation preserving. The image $\pi(\text{Emb}(S, M))$ is the manifold $\text{Gr}_k^S(M)$ of k -dimensional submanifolds of M of type S . Then $\pi : \text{Emb}(S, M) \rightarrow \text{Gr}_k^S(M)$ is a principal bundle over $\text{Gr}_k^S(M)$ with structure group $\text{Diff}_+(S)$, the group of orientation preserving diffeomorphisms of S .

Note that there is a natural action of the group $\text{Diff}(M)$ on the non-linear Grassmannian $\text{Gr}_k(M)$ given by $\varphi \cdot N = \varphi(N)$. Let $\tilde{\varphi}$ be the diffeomorphism

of $\text{Gr}_k(M)$ induced by the action of $\varphi \in \text{Diff}(M)$. Then $\tilde{\varphi} \circ \pi = \pi \circ \bar{\varphi}$ for the restriction of $\bar{\varphi}(f) = \varphi \circ f$ to a diffeomorphism of $\text{Emb}(S, M) \subset \mathcal{F}(S, M)$. As a consequence, the infinitesimal generators for the $\text{Diff}(M)$ actions on $\text{Gr}_k(M)$ and on $\text{Emb}(S, M)$ are π -related. This means that for all $X \in \mathfrak{X}(M)$, the vector fields \tilde{X} on $\text{Gr}_k(M)$ given by $\tilde{X}(N) = X|_N$ and \bar{X} on $\text{Emb}(S, M)$ given by $\bar{X}(f) = X \circ f$ are π -related.

Proposition 10. *The hat map on $\text{Emb}(S, M)$ and the tilda map on $\text{Gr}_k(M)$ are related by $\widehat{\omega} = \pi^*\tilde{\omega}$, for any k -dimensional oriented manifold S .*

Proof. For the proof we use the fact that $\mathfrak{X}(M)$ acts infinitesimally transitive on $\text{Emb}(S, M)$, so $T_f \text{Emb}(S, M) = \{X \circ f : X \in \mathfrak{X}(M)\}$. With (9) and (11) we compute:

$$\begin{aligned} (\pi^*\tilde{\omega})_f(X_1 \circ f, \dots, X_{p-k} \circ f) &= \tilde{\omega}_{f(S)}(X_1|_{f(S)}, \dots, X_{p-k}|_{f(S)}) \\ &= \int_{f(S)} i_{X_{p-k}} \dots i_{X_1} \omega = \int_S f^*(i_{X_{p-k}} \dots i_{X_1} \omega) = \widehat{\omega}_f(X_1 \circ f, \dots, X_{p-k} \circ f), \end{aligned}$$

since \bar{X} and \tilde{X} are π -related. □

From the properties of the hat pairing presented in proposition 5, lemma 6 and theorem 1, a hat calculus follows easily:

Proposition 11. *For any $\omega \in \Omega^p(M)$, $\varphi \in \text{Diff}(M)$, $X \in \mathfrak{X}(M)$, and $\eta \in \mathcal{F}(M', M)$ with push-forward $\bar{\eta} : \mathcal{F}(S, M') \rightarrow \mathcal{F}(S, M)$, the following identities hold:*

1. $\bar{\varphi}^*\widehat{\omega} = \widehat{\varphi^*\omega}$ and $\bar{\eta}^*\widehat{\omega} = \widehat{\eta^*\omega}$
2. $L_{\bar{X}}\widehat{\omega} = \widehat{L_X\omega}$
3. $i_{\bar{X}}\widehat{\omega} = \widehat{i_X\omega}$
4. $\mathbf{d}\widehat{\omega} = \widehat{\mathbf{d}\omega}$.

Remark 12. If S is a manifold with boundary, then the formula 4. above receives an extra term coming from integration over the boundary ∂S as in proposition 4:

$$\mathbf{d}\widehat{\omega} = \widehat{\mathbf{d}\omega} + (-1)^{p-k} r_{\partial}^* \widehat{\omega}^{\partial} \quad (12)$$

for $\omega \in \Omega^p(M)$. As before, $r_{\partial} : \mathcal{F}(S, M) \rightarrow \mathcal{F}(\partial S, M)$ denotes the restriction map on functions and $\omega \in \Omega^p(M) \mapsto \widehat{\omega}^{\partial} \in \Omega^{p-k+1}(\mathcal{F}(\partial S, M))$.

Now the properties of the tilda calculus follow immediately from proposition 11.

Proposition 13. *[HV04] For any $\omega \in \Omega^p(M)$, $\varphi \in \text{Diff}(M)$ and $X \in \mathfrak{X}(M)$, the following identities hold:*

1. $\tilde{\varphi}^*\tilde{\omega} = \widetilde{\varphi^*\omega}$
2. $L_{\tilde{X}}\tilde{\omega} = \widetilde{L_X\omega}$
3. $i_{\tilde{X}}\tilde{\omega} = \widetilde{i_X\omega}$
4. $\mathbf{d}\tilde{\omega} = \widetilde{\mathbf{d}\omega}$.

Proof. We verify the identities 1. and 4. From relation 1. from proposition 11 we get that

$$\pi^*\tilde{\varphi}^*\tilde{\omega} = \tilde{\varphi}^*\pi^*\tilde{\omega} = \tilde{\varphi}^*\widehat{\omega} = \widehat{\varphi^*\omega} = \pi^*\widehat{\varphi^*\omega},$$

and this implies the first identity. Using identity 4. from proposition 11 we compute

$$\pi^*\mathbf{d}\tilde{\omega} = \mathbf{d}\pi^*\tilde{\omega} = \mathbf{d}\widehat{\omega} = \widehat{\mathbf{d}\omega} = \pi^*\widehat{\mathbf{d}\omega},$$

which shows the last identity. \square

Hamiltonian formalism for p -branes

In this section we show how the hat calculus appears in the hamiltonian formalism for p -branes and open p -branes [AS05] [BZ05].

Let S be a compact oriented p -dimensional manifold. The phase space for the p -brane world volume $S \times \mathbb{R}$ is the cotangent bundle $T^*\mathcal{F}(S, M)$, where the canonical symplectic form is twisted. The twisting consists in adding a magnetic term, namely the pull-back of a closed 2-form on the base manifold, to the canonical symplectic form on a cotangent bundle [MR99]. These twisted symplectic forms appear also in cotangent bundle reduction.

We consider a closed differential form $H \in \Omega^{p+2}(M)$. Since $\dim S = p$, the hat map (10) provides a closed 2-form \widehat{H} on $\mathcal{F}(S, M)$. If $\pi_{\mathcal{F}} : T^*\mathcal{F}(S, M) \rightarrow \mathcal{F}(S, M)$ denotes the canonical projection, the twisted symplectic form on $T^*\mathcal{F}(S, M)$ is

$$\Omega_H = -\mathbf{d}\Theta_{\mathcal{F}} + \frac{1}{2}\pi_{\mathcal{F}}^*\widehat{H},$$

where $\Theta_{\mathcal{F}}$ is the canonical 1-form on $T^*\mathcal{F}(S, M)$.

For the description of open branes one considers a compact oriented p -dimensional manifold S with boundary ∂S and a submanifold D of M . The phase space is in this case the cotangent bundle $T^*\mathcal{F}_D(S, M)$ over the manifold [M80]

$$\mathcal{F}_D(S, M) = \{f : S \rightarrow M \mid f(\partial S) \subset D\}.$$

The twisting of the canonical symplectic form is done with a closed differential form $H \in \Omega^{p+2}(M)$ with $i^*H = \mathbf{d}B$ for some $B \in \Omega^{p+1}(D)$, where $i : D \rightarrow M$ denotes the inclusion. The twisted symplectic form on $T^*\mathcal{F}_D(S, M)$ is

$$\Omega_{(H,B)} = -\mathbf{d}\Theta_{\mathcal{F}_D} + \frac{1}{2}\pi_{\mathcal{F}_D}^*(\widehat{H} - \partial^*\widehat{B}^\partial)$$

with $\partial : \mathcal{F}_D(S, M) \rightarrow \mathcal{F}(\partial S, D)$ the restriction map and $\pi_{\mathcal{F}_D} : T^*\mathcal{F}_D(S, M) \rightarrow \mathcal{F}_D(S, M)$. To distinguish between the hat calculus for $\mathcal{F}(S, M)$ and the hat calculus for $\mathcal{F}(\partial S, M)$, we denote $\widehat{\cdot}^\partial : \Omega^n(M) \rightarrow \Omega^{n-p+1}(\mathcal{F}(\partial S, M))$.

The only thing we have to verify is the closedness of $\widehat{H} - \partial^*\widehat{B}^\partial$. We first notice that (12) implies $\mathbf{d}\widehat{H} = \widehat{\mathbf{d}H} + r_\partial^*\widehat{H}^\partial$, where $r_\partial : \mathcal{F}(S, M) \rightarrow \mathcal{F}(\partial S, M)$ denotes the restriction map, and identity 4 from proposition 11 implies $\widehat{\mathbf{d}B}^\partial = \mathbf{d}\widehat{B}^\partial$. On the other hand identity 1 from proposition 11 ensures that $\widehat{i^*H}^\partial = \bar{i}^*\widehat{H}^\partial$, with $\bar{i} : \mathcal{F}(\partial S, D) \rightarrow \mathcal{F}(\partial S, M)$ denoting the push-forward by $i : D \rightarrow M$. Knowing that $r_\partial = \bar{i} \circ \partial$, we compute:

$$\mathbf{d}\widehat{H} = \widehat{\mathbf{d}H} + r_\partial^*\widehat{H}^\partial = \partial^*\bar{i}^*\widehat{H}^\partial = \partial^*\widehat{i^*H}^\partial = \partial^*\widehat{\mathbf{d}B}^\partial = \mathbf{d}\partial^*\widehat{B}^\partial,$$

so the closed 2-form $\widehat{H} - \partial^*\widehat{B}^\partial$ provides a twist for the canonical symplectic form on the cotangent bundle $T^*\mathcal{F}_D(S, M)$.

Non-linear Grassmannians as symplectic manifolds

In this subsection we recall properties of the co-dimension two non-linear Grassmannian as a symplectic manifold.

Proposition 14. [I96] *Let M be a closed m -dimensional manifold with volume form ν . The tilda map provides a symplectic form $\tilde{\nu}$ on $\text{Gr}_{m-2}(M)$*

$$\tilde{\nu}_N(\tilde{X}_N, \tilde{Y}_N) = \int_N i_{Y_N} i_{X_N} \nu,$$

for \tilde{X}_N and \tilde{Y}_N sections of TN^\perp determined by sections X_N and Y_N of $TM|_N$.

Proof. The 2-form $\tilde{\nu}$ is closed since $\mathbf{d}\tilde{\nu} = \widetilde{\mathbf{d}\nu}$ by the tilda calculus. To verify that it is also (weakly) non-degenerate, let X_N be an arbitrary vector field along N such that $\int_N i_{Y_N} i_{X_N} \nu = 0$ for all vector fields Y_N along N . Then X_N must be tangent to N , so $\tilde{X}_N = 0$. \square

In dimension $m = 3$ the symplectic form $\tilde{\nu}$ is known as the Marsden–Weinstein symplectic form on the space of unparameterized oriented links, see [MW83] [B93].

Hamiltonian $\text{Diff}_{ex}(M, \nu)$ action. The action of the group $\text{Diff}(M, \nu)$ of volume preserving diffeomorphisms of M on $\text{Gr}_{m-2}(M)$ preserves the symplectic form $\tilde{\nu}$:

$$\tilde{\varphi}^*\tilde{\nu} = \widetilde{\varphi^*\nu} = \tilde{\nu}, \quad \forall \varphi \in \text{Diff}(M, \nu).$$

The subgroup $\text{Diff}_{ex}(M, \nu)$ of exact volume preserving diffeomorphisms acts in a hamiltonian way on the symplectic manifold $(\text{Gr}_{m-2}(M), \tilde{\nu})$. Its Lie algebra is $\mathfrak{X}_{ex}(M, \nu)$, the Lie algebra of exact divergence free vector fields, i.e. vector fields

X_α such that $i_{X_\alpha}\nu = \mathbf{d}\alpha$ for a potential form $\alpha \in \Omega^{m-2}(M)$. The infinitesimal action of X_α is the vector field \tilde{X}_α . By the tilda calculus $\tilde{\alpha} \in \mathcal{F}(\text{Gr}_{m-2}(M))$ is a hamiltonian function for the hamiltonian vector field \tilde{X}_α :

$$i_{\tilde{X}_\alpha}\tilde{\nu} = \widetilde{i_{X_\alpha}\nu} = \widetilde{\mathbf{d}\alpha} = \mathbf{d}\tilde{\alpha}.$$

It depends on the particular choice of the potential α of X_α . A fixed continuous right inverse $b : \mathbf{d}\Omega^{m-2}(M) \rightarrow \Omega^{m-2}(M)$ to the differential \mathbf{d} picks up a potential $b(\mathbf{d}\alpha)$ of X_α . The corresponding momentum map is:

$$\mathbf{J} : \mathcal{M} \rightarrow \mathfrak{X}_{ex}(M, \nu)^*, \quad \langle \mathbf{J}(N), X_\alpha \rangle = \widetilde{b(\mathbf{d}\alpha)}(N) = \int_N b(\mathbf{d}\alpha).$$

On the connected component \mathcal{M} of $N \in \text{Gr}_{m-2}(M)$, the non-equivariance of \mathbf{J} is measured by the Lie algebra 2-cocycle on $\mathfrak{X}_{ex}(M, \nu)$

$$\begin{aligned} \sigma_N(X, Y) &= \langle \mathbf{J}(N), [X, Y]^{op} \rangle - \tilde{\nu}(\tilde{X}, \tilde{Y})(N) = (\widetilde{b\mathbf{d}i_Y i_X \nu})(N) - (\widetilde{i_Y i_X \nu})(N) \\ &= (\widetilde{Pi_X i_Y \nu})(N) = \int_N Pi_X i_Y \nu. \end{aligned}$$

Here $P = 1_{\Omega^{m-2}(M)} - b \circ \mathbf{d}$ is a continuous linear projection on the subspace of closed $(m-2)$ -forms and $(X, Y) \mapsto [Pi_Y i_X \nu] \in H^{m-2}(M)$ is the universal Lie algebra 2-cocycle on $\mathfrak{X}_{ex}(M, \nu)$ [R95]. The cocycle σ_N is cohomologous to the Lichnerowicz cocycle

$$\sigma_\eta(X, Y) = \int_M \eta(X, Y)\nu, \quad (13)$$

where η is a closed 2-form Poincaré dual to N [V09].

If ν is an integral volume form, then σ_N is integrable [I96]. The connected component \mathcal{M} of $\text{Gr}_{m-2}(M)$ is a coadjoint orbit of a 1-dimensional central Lie group extension of $\text{Diff}_{ex}(M, \nu)$ integrating σ_N , and $\tilde{\nu}$ is the Kostant-Kirillov-Souriau symplectic form. [HV04].

4 Bar map

When a volume form μ on the compact k -dimensional manifold S is given, one can associate to each differential p -form on M a differential p -form on $\mathcal{F}(S, M)$

$$\bar{\omega}(Y_f^1, \dots, Y_f^p) = \int_S \omega(Y_f^1, \dots, Y_f^p)\mu, \quad \forall Y_f^i \in T_f \mathcal{F}(S, M),$$

where $\omega(Y_f^1, \dots, Y_f^p) : x \mapsto \omega_{f(x)}(Y_f^1(x), \dots, Y_f^p(x))$ defines a smooth function on S . In this way a *bar map* is defined. Formula (2) assures that this bar map is just the hat pairing of differential forms on M with the volume form μ

$$\bar{\omega} = \widehat{\omega \cdot \mu} = \int_S \text{ev}^* \omega \wedge \text{pr}^* \mu. \quad (14)$$

From the properties of the hat pairing presented in proposition 5, lemma 6 and theorem 1, one can develop a bar calculus.

Proposition 15. *For any $\omega \in \Omega^p(M)$, $\varphi \in \text{Diff}(M)$ and $X \in \mathfrak{X}(M)$, the following identities hold:*

1. $\bar{\varphi}^* \bar{\omega} = \overline{\varphi^* \omega}$
2. $L_{\bar{X}} \bar{\omega} = \overline{L_X \omega}$
3. $i_{\bar{X}} \bar{\omega} = \overline{i_X \omega}$
4. $\mathbf{d} \bar{\omega} = \overline{\mathbf{d} \omega}$.

$\mathcal{F}(S, M)$ as symplectic manifold

Let (M, ω) be a connected symplectic manifold and S a compact k -dimensional manifold with a fixed volume form μ , normalized such that $\int_S \mu = 1$. The following fact is well known:

Proposition 16. *The bar map provides a symplectic form $\bar{\omega}$ on $\mathcal{F}(S, M)$:*

$$\bar{\omega}_f(X_f, Y_f) = \int_S \omega(X_f, Y_f) \mu.$$

Proof. That $\bar{\omega}$ is closed follows from the bar calculus: $\mathbf{d} \bar{\omega} = \overline{\mathbf{d} \omega} = 0$. The (weakly) non-degeneracy of $\bar{\omega}$ can be verified as follows. If the vector field X_f on M along S is non-zero, then $X_f(x) \neq 0$ for some $x \in S$. Because ω is non-degenerate, one can find another vector field Y_f along f such that $\omega(X_f, Y_f)$ is a bump function on S . Then $\bar{\omega}(X_f, Y_f) = \int_S \omega(X_f, Y_f) \mu \neq 0$, so X_f does not belong to the kernel of $\bar{\omega}$, thus showing that the kernel of $\bar{\omega}$ is trivial. \square

Hamiltonian action on M . Let G be a Lie group acting in a hamiltonian way on M with momentum map $J : M \rightarrow \mathfrak{g}^*$. Then $\mathcal{F}(S, M)$ inherits a G -action: $(g \cdot f)(x) = g \cdot (f(x))$ for any $x \in S$. The infinitesimal generator is $\xi_{\mathcal{F}} = \bar{\xi}_M$ for any $\xi \in \mathfrak{g}$, where ξ_M denotes the infinitesimal generator for the G -action on M . The bar calculus shows quickly that G acts in a hamiltonian way on $\mathcal{F}(S, M)$ with momentum map

$$\mathbf{J} = \bar{J} : \mathcal{F}(S, M) \rightarrow \mathfrak{g}^*, \quad \bar{J}(f) = \int_S (J \circ f) \mu, \quad \forall f \in \mathcal{F}(S, M).$$

Indeed, for all $\xi \in \mathfrak{g}$

$$i_{\xi_{\mathcal{F}}} \bar{\omega} = i_{\bar{\xi}_M} \bar{\omega} = \overline{i_{\xi_M} \omega} = \overline{\mathbf{d} \langle J, \xi \rangle} = \mathbf{d} \langle \bar{J}, \xi \rangle.$$

Let M be connected and let σ be the \mathbb{R} -valued Lie algebra 2-cocycle on \mathfrak{g} measuring the non-equivariance of J , i.e.

$$\sigma(\xi, \eta) = \langle J(x), [\xi, \eta] \rangle - \omega(\xi_M, \eta_M)(x), \quad x \in M,$$

(both terms are hamiltonian function for the vector field $[\xi, \eta]_M = -[\xi_M, \eta_M]$). Then the non-equivariance of $\mathbf{J} = \bar{J}$ is also measured by σ : for all $f \in \mathcal{F}(S, M)$

$$\langle \bar{J}(f), [\xi, \eta] \rangle - \bar{\omega}(\xi_{\mathcal{F}}, \eta_{\mathcal{F}})(f) = \overline{\langle J, [\xi, \eta] \rangle}(f) - \overline{\omega(\xi_M, \eta_M)}(f) = \sigma(\xi, \eta).$$

Hamiltonian $\text{Diff}_{ham}(M, \omega)$ action. The action of the group $\text{Diff}(M, \omega)$ of symplectic diffeomorphisms preserves the symplectic form $\bar{\omega}$:

$$\bar{\varphi}^* \bar{\omega} = \overline{\varphi^* \omega} = \bar{\omega}, \quad \forall \varphi \in \text{Diff}(M, \omega).$$

The subgroup $\text{Diff}_{ham}(M, \omega)$ of hamiltonian diffeomorphisms of M acts in a hamiltonian way on the symplectic manifold $\mathcal{F}(S, M)$. The infinitesimal action of $X_h \in \mathfrak{X}_{ham}(M, \omega)$, $h \in \mathcal{F}(M)$, is the hamiltonian vector field \bar{X}_h on $\mathcal{F}(S, M)$ with hamiltonian function \bar{h} . This follows by the bar calculus:

$$\mathbf{d}\bar{h} = \overline{\mathbf{d}h} = \overline{i_{X_h} \omega} = i_{\bar{X}_h} \bar{\omega}.$$

The hamiltonian function \bar{h} of \bar{X}_h depends on the particular choice of the hamiltonian function h . To solve this problem we fix a point $x_0 \in M$ and we choose the unique hamiltonian function h with $h(x_0) = 0$, since M is connected. The corresponding momentum map is

$$\mathbf{J} : \mathcal{F}(S, M) \rightarrow \mathfrak{X}_{ham}(M, \omega)^*, \quad \langle \mathbf{J}(f), X_h \rangle = \bar{h}(f) = \int_S (h \circ f) \mu.$$

The Lie algebra 2-cocycle on $\mathfrak{X}_{ham}(M, \omega)$ measuring the non-equivariance of the momentum map is

$$\sigma(X, Y) = -\omega(X, Y)(x_0),$$

by the bar calculus

$$\begin{aligned} \sigma(X, Y)(f) &= \langle \mathbf{J}(f), [X, Y]^{op} \rangle - \bar{\omega}(X_{\mathcal{F}}, Y_{\mathcal{F}})(f) \\ &= \overline{\omega(X, Y) - \omega(X, Y)(x_0)}(f) - \bar{\omega}(\bar{X}, \bar{Y})(f) = -\omega(X, Y)(x_0). \end{aligned}$$

This is a Lie algebra cocycle describing the central extension

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{F}(M) \rightarrow \mathfrak{X}_{ham}(M, \omega) \rightarrow 0$$

where $\mathcal{F}(M)$ is endowed with the canonical Poisson bracket. A group cocycle on $\text{Diff}_{ham}(M, \omega)$ integrating the Lie algebra cocycle σ if ω exact is studied in [ILM06].

Hamiltonian $\text{Diff}_{ex}(S, \mu)$ action. The (left) action of the group $\text{Diff}(S, \mu)$ of volume preserving diffeomorphisms preserves the symplectic form $\bar{\omega}$:

$$\widehat{\psi^* \bar{\omega}} = \widehat{\psi^* \omega \cdot \mu} = \widehat{\omega \cdot \psi^* \mu} = \widehat{\omega \cdot \mu} = \bar{\omega}, \quad \forall \psi \in \text{Diff}(S, \mu).$$

The subgroup $\text{Diff}_{ex}(S, \mu)$ of exact volume preserving diffeomorphisms acts in a hamiltonian way on the symplectic manifold $\mathcal{F}(S, M)$. The infinitesimal action of the exact divergence free vector field $X_\alpha \in \mathfrak{X}_{ex}(S, \mu)$ with potential form $\alpha \in \Omega^{k-2}(S)$ is the hamiltonian vector field \widehat{X}_α on $\mathcal{F}(S, M)$ with hamiltonian function $\widehat{\omega \cdot \alpha}$. Indeed, from $i_{X_\alpha} \mu = \mathbf{d}\alpha$ follows by the hat calculus that

$$\mathbf{d}(\widehat{\omega \cdot \alpha}) = \widehat{\mathbf{d}\omega \cdot \alpha} + \widehat{\omega \cdot \mathbf{d}\alpha} = \widehat{\omega \cdot i_{X_\alpha} \mu} = i_{\widehat{X}_\alpha} \widehat{\omega \cdot \mu} = i_{\widehat{X}_\alpha} \bar{\omega}.$$

If the symplectic form ω is exact, then the corresponding momentum map is

$$\mathbf{J} : \mathcal{F}(S, M) \rightarrow \mathfrak{X}_{ex}(S, \mu)^*, \quad \langle \mathbf{J}(f), X_\alpha \rangle = \widehat{(\omega \cdot \alpha)}(f) = \int_S f^* \omega \wedge \alpha.$$

It takes values in the regular part of $\mathfrak{X}_{ex}(S, \mu)^*$, which can be identified with $\mathbf{d}\Omega^1(S)$, so we can write $\mathbf{J}(f) = f^* \omega$ under this identification.

In general the hamiltonian function $\widehat{\omega \cdot \alpha}$ of \widehat{X}_α depends on the particular choice of the potential form α of X_α . To fix this problem we consider as in Section 3 a continuous right inverse $b : \mathbf{d}\Omega^{m-2}(M) \rightarrow \Omega^{m-2}(M)$ to the differential \mathbf{d} , so $b(\mathbf{d}\alpha)$ is a potential for X_α . The corresponding momentum map is

$$\mathbf{J} : \mathcal{F}(S, M) \rightarrow \mathfrak{X}_{ex}(S, \mu)^*, \quad \langle \mathbf{J}(f), X_\alpha \rangle = \widehat{(\omega \cdot b\mathbf{d}\alpha)}(f) = \int_S f^* \omega \wedge b(\mathbf{d}\alpha).$$

On a connected component \mathcal{F} of $\mathcal{F}(S, M)$, the non-equivariance of \mathbf{J} is measured by the Lie algebra 2-cocycle

$$\begin{aligned} \sigma_{\mathcal{F}}(X, Y) &= \langle \mathbf{J}(f), [X, Y] \rangle - \bar{\omega}(\widehat{X}, \widehat{Y})(f) = (\omega \cdot b\mathbf{d}i_Y i_X \mu)^\wedge(f) - (\omega \cdot i_Y i_X \mu)^\wedge(f) \\ &= (\omega \cdot P i_X i_Y \mu)^\wedge(f) = \int_S f^* \omega \wedge P i_X i_Y \mu \end{aligned}$$

on the Lie algebra of exact divergence free vector fields, for $P = 1 - b\mathbf{d}$ the projection on the subspace of closed $(m-2)$ -forms. It does not depend on $f \in \mathcal{F}$, because the cohomology class $[f^* \omega] \in H^2(S)$ does not depend on the choice of f . The cocycle $\sigma_{\mathcal{F}}$ is cohomologous to the Lichnerowicz cocycle $\sigma_{f^* \omega}$ defined in (13) [V09]. Since $\int_S \mu = 1$, the cocycle $\sigma_{\mathcal{F}}$ is integrable if and only if the cohomology class of $f^* \omega$ is integral [I96].

Remark 17. The two equivariant momentum maps on the symplectic manifold $\mathcal{F}(S, M)$, for suitable central extensions of the hamiltonian group $\text{Diff}_{ham}(M, \omega)$ and of the group $\text{Diff}_{ex}(S, \mu)$ of exact volume preserving diffeomorphisms, form the dual pair for ideal incompressible fluid flow [MW83] [GBV09].

5 Appendix: Fiber integration

Chapter VII in [GHV72] is devoted to the concept of integration over the fiber in locally trivial bundles. We particularize this fiber integration to the case of trivial bundles $S \times M \rightarrow M$, listing its main properties without proofs.

Let S be a compact k -dimensional manifold. Fiber integration over S assigns to $\omega \in \Omega^n(S \times M)$ the differential form $\int_S \omega \in \Omega^{n-k}(M)$ defined by

$$\left(\int_S \omega\right)(x) = \int_S \omega_x \in \Lambda^{n-k} T_x^* M, \quad \forall x \in M,$$

where $\omega_x \in \Omega^k(S, \Lambda^{n-k} T_x^* M)$ is the retrenchment of ω to the fiber over x :

$$\langle \omega_x(Z_s^1, \dots, Z_s^{n-k}), X_x^1 \wedge \dots \wedge X_x^k \rangle = \omega_{(s,x)}(X_x^1, \dots, X_x^k, Z_s^1, \dots, Z_s^{n-k})$$

for all $X_x^i \in T_x M$ and $Z_s^j \in T_s S$.

The properties of the fiber integration used in the text are special cases of the propositions (VIII) and (X) in [GHV72]:

1. Pull-back of fiber integrals:

$$f^* \int_S \omega = \int_S (1_S \times f)^* \omega, \quad \forall f \in \mathcal{F}(M', M), \quad (15)$$

with infinitesimal version

$$L_X \int_S \omega = \int_S L_{0_S \times X} \omega, \quad \forall X \in \mathfrak{X}(M). \quad (16)$$

2. Invariance under pull-back by orientation preserving diffeomorphisms of S :

$$\int_S (\varphi \times 1_M)^* \omega = \int_S \omega, \quad \forall \varphi \in \text{Diff}_+(S), \quad (17)$$

with infinitesimal version $\int_S L_{Z \times 0_M} \omega = 0, \quad \forall Z \in \mathfrak{X}(S)$.

3. Insertion of vector fields into fiber integrals:

$$i_X \int_S \omega = \int_S i_{0_S \times X} \omega, \quad \forall X \in \mathfrak{X}(M). \quad (18)$$

4. Integration along boundary free manifolds commutes with differentiation. When ∂S denotes the boundary of the k -dimensional compact manifold S and $i_\partial : \partial S \rightarrow S$ the inclusion,

$$\mathbf{d} \int_S \beta - \int_S \mathbf{d} \beta = (-1)^{n-k} \int_{\partial S} (i_\partial \times 1_M)^* \beta \quad (19)$$

holds for any differential n -form β on $S \times M$.

References

- [AS05] A. Alekseev and T. Strobl, *Current algebras and differential geometry*, arXiv:hep-th/0410183v2, 2005.
- [BZ05] G. Bonelli and M. Zabzine, *From current algebras for p -branes to topological M -theory*, arXiv:hep-th/0507051v2, 2005.
- [B93] J.-L. Brylinski, *Loop spaces, characteristic classes and geometric quantization*, Progress in Math. 107, Birkhäuser, 1993.
- [GBV09] F. Gay-Balmaz and C. Vizman, *Dual pairs in fluid dynamics*, preprint 2009.
- [GHV72] W. Greub, S. Halperin and R. Vanstone, *Connections, curvature and cohomology*, Vol. I, Pure and Applied Mathematics 47, Academic Press, New York-London, 1972.
- [HV04] S. Haller and C. Vizman, *Non-linear Grassmannians as coadjoint orbits*, Math. Ann. 329 (2004), 771–785.
- [H76] M.W. Hirsch, *Differential topology*, Graduate Texts in Math. 33, Springer, 1976.
- [I96] R.S. Ismagilov, *Representations of infinite-dimensional groups*, Translations of Mathematical Monographs 152, American Mathematical Society, Providence, RI, 1996.
- [ILM06] R.S. Ismagilov, M. Losik, and P.W. Michor, *A 2-cocycle on a group of symplectomorphisms*, Moscow Math. J. 6 (2006), 307–315.
- [KM97] A. Kriegl and P.W. Michor, *The convenient setting of global analysis*, Mathematical Surveys and Monographs 53, American Mathematical Society, Providence, RI, 1997.
- [MW83] J.E. Marsden and A. Weinstein, *Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids*. Phys. D 7 (1983), 305–323.
- [MR99] J.E. Marsden and T. Ratiu, *Introduction to Mechanics and Symmetry*, 2nd edition, Springer, 1999.
- [M80] P.W. Michor, *Manifolds of differentiable mappings* Shiva Mathematics Series 3, Shiva Publ., Orpington, 1980.
- [R95] Roger, C., *Extensions centrales d’algèbres et de groupes de Lie de dimension infinie, algèbre de Virasoro et généralisations*, Rep. Math. Phys., 35 (1995) 225–266.

- [V09] C. Vizman, *Lichnerowicz cocycles and central Lie group extensions*, Preprint, 2009.